

Quantum particle on a Möbius strip, coherent states and projection operators

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(Date textdate; Received textdate; Revised textdate; Accepted textdate; Published textdate)

Abstract

The coherent states for a quantum particle on a Möbius strip are constructed and their relation with the natural phase space for fermionic fields is shown. The explicit comparison of the obtained states with previous works where the cylinder quantization was used and the spin $1/2$ was introduced by hand is given, and the relation between the geometrical phase space, constraints and projection operators is analyzed and discussed.

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I. INTRODUCTION

Coherent States have attracted much attention in many branches of physics [1]. In spite of their importance, the theory of CS when the configuration space has non trivial topology is far from complete. CS for a quantum particle on a circle [3] and a sphere have been introduced very recently, and also in the case of torus [4,7]. If well , in all these works the different constructions of the CS for the boson case is practically straightforward, the simple addition by hand of $1/2$ to the angular momentum operator J for the fermionic case into the corresponding CS remains obscure and non-natural. The question that naturally arises is: there exists any geometry for the phase space in which the CS construction leads precisely a fermionic quantization condition? Recently in previous works [9] we demonstrate the positive answer to this question showing that the CS for a quantum particle on the Möbius strip geometry is the natural candidate to describe fermions exactly as the cylinder geometry for bosons. Then, the purpose of this paper is to analyze deeply this relation between

the coherent states and the geometry of the physical phase space taking into account two important roles playing by the CS: as projector operators [10] and as the main link between classical and quantum formulations of a given system [5].

II. REQUIREMENTS FOR THE COHERENT STATES

It is well known that coherent states provide naturally a close connection between classical and quantum formulations of a given system. A suitable set of requirements for these states is given, in association with a specific Hamiltonian operator \mathcal{H} , by

- (a) Continuity: $(J', \gamma') \rightarrow (J, \gamma) \Rightarrow |J', \gamma'\rangle \rightarrow |J, \gamma\rangle$.
- (b) Resolution of the unity: $\mathbb{I} = \int |J, \gamma\rangle \langle J, \gamma| d\mu(J, \gamma)$.
- (c) Temporal stability: $e^{-i\mathcal{H}t} |J, \gamma\rangle = |J, \gamma + \omega t\rangle, \omega = \text{constant}$.
- (d) Action identity: $\langle J, \gamma | \mathcal{H} | J, \gamma \rangle = \omega J$.

The first two requirements emphasize the fact that the identity operator may be understood in a restricted sense, namely as a projector onto a finite or infinite subspace. The third requirement ensures that the time evolution of any coherent state is always a coherent state. As was showed clearly by Gazeau and Klauder in [4], in this evolution, J remains constant while γ increases linearly. These properties are similar to the classical behavior of action-angle variables. If J and γ denote canonical action-angle variables, they would enter the classical action in the following form

$$I = \int_0^T (J\dot{\gamma} - \omega J) dt.$$

As is easily seen, the classical action can be viewed as the restricted evaluation of the quantum action functional:

$$I = \int_0^T \left[i \langle J, \gamma | \frac{d}{dt} | J, \gamma \rangle - \langle J, \gamma | \mathcal{H} | J, \gamma \rangle \right] dt.$$

for different paths $\{|J(t), \gamma(t)\rangle : 0 \leq t \leq T\}$ lying in a two dimensional manifold in Hilbert space. Thus the fourth requirement simply codifies the fact that the two coordinates (J, γ) are canonical action-angle variables (it will follow that the kinematical term is $J\dot{\gamma}$ as needed [5]). At this point seems to be necessary to make the following observations: firstly, the physical meaning of the third requirement is to assert that the path in the Hilbert space represented by $\{|J, \gamma + \omega t\rangle : 0 \leq t \leq T\}$ is actually the true quantum temporal for the

quantum Hamiltonian \mathcal{H} . Then, the restricted quantum action functional in this case is *exact*, see [1] - Gazeau and references therein; a wider set of variational paths starting at $|J, \gamma\rangle$ at $t = 0$ leads to the same extreme path. Secondly, it is well known that the lack of uniqueness in the possible families of CS corresponding to a given Hamiltonian with discrete spectrum is because the fourth requirement was not taken into account[1]

III. GEOMETRY OF THE MÖBIUS BAND AND DYNAMICS

The position of a point into the Möbius strip geometry can be parameterized as

$$P_0 = (X_0, Y_0, Z_0) , P_1 = (X_0 + X_1, Y_0 + Y_1, Z_0 + Z_1) \quad (1)$$

The coordinates of P_0 describes the central cylinder (generated by the invariant fiber of the middle of the weight of the strip)

$$Z_0 = l , \quad X_0 = R \cos\varphi, \quad Y_0 = R \sin\varphi, \quad (2)$$

(this is topological invariant of the geometry under study).

The coordinates of P_1 (the boundaries of the Möbius band) are of P_0 (the cylinder) plus

$$Z_1 = r \cos\theta, \quad X_1 = r \sin\theta \cos\varphi, \quad Y_1 = r \sin\theta \sin\varphi, \quad (3)$$

The weight of the band is obviously $2r$, then our space of phase is embedded into of the Torus

$$\begin{aligned} X &= R \cos\varphi + r \sin\theta \cos\varphi \\ Y &= R \sin\varphi + r \sin\theta \sin\varphi \\ Z &= l + r \cos\theta \end{aligned} \quad (4)$$

The important point is that the angles are not independent in the case of the Möbius band and are related by the following constraint

$$\theta = \frac{\varphi + \pi}{2} \quad (5)$$

It is very important, this constraint effectively reduces the degree of freedom from the torus to the unoriented surface.

In order to study the dynamics in this non-trivial geometry, we construct the non-relativistic Lagrangian

$$L = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) \quad (6)$$

$$L = \frac{1}{2} \left\{ \dot{\varphi}^2 \left[(1 + r \cos(\varphi/2))^2 + \frac{r^2}{4} \right] - r \cos(\varphi/2) \dot{Z}_0 \dot{\varphi} + (\dot{Z}_0)^2 \right\} \quad (7)$$

From the above expression the equations of motion are

$$\frac{\partial L}{\partial \dot{\varphi}} = \dot{\varphi} \left[(1 + r \cos(\varphi/2))^2 + \frac{r^2}{4} \right] - \frac{r}{2} \cos(\varphi/2) \dot{Z}_0 \quad (8)$$

$$\frac{\partial L}{\partial \dot{Z}_0} = -\frac{r}{2} \cos(\varphi/2) \dot{\varphi} + \dot{Z}_0 \quad (9)$$

$$\frac{\partial L}{\partial \varphi} = \frac{r}{2} \sin(\varphi/2) \dot{\varphi} \left[-\dot{\varphi} (1 + r \cos(\varphi/2)) + \frac{\dot{Z}}{2} \right] \quad (10)$$

$$\frac{\partial L}{\partial Z_0} = 0 \quad (11)$$

Taking account that Z_0 is a cyclic coordinate, we have the following constraint

$$\left(\frac{\partial L}{\partial \dot{Z}_0} \right) - \frac{\partial L}{\partial Z_0} = 0 \Rightarrow \frac{\partial L}{\partial \dot{Z}_0} = L_0 = -\frac{r}{2} \cos(\varphi/2) \dot{\varphi} + \dot{Z}_0 \quad (12)$$

then, looking for the dynamical expressions for $\dot{\varphi}$

$$\frac{\partial L}{\partial \dot{\varphi}} = \dot{\varphi} \left[(1 + r \cos(\varphi/2))^2 + \frac{r^2}{4} \sin^2(\varphi/2) \right] - \frac{r}{2} \cos(\varphi/2) L_0 = J \quad (13)$$

From the Lagrangian (7) the Hamiltonian is not difficult to obtain

$$\begin{aligned} H &= p_\varphi \dot{\varphi} + p_{z_0} \dot{Z}_0 - L \\ &= \frac{1}{2} \left\{ \dot{\varphi}^2 \left[(1 + r \cos(\varphi/2))^2 + \frac{r^2}{4} \right] - r \cos(\varphi/2) \dot{Z}_0 \dot{\varphi} + (\dot{Z}_0)^2 \right\} = L \end{aligned} \quad (14)$$

that trough the constraint (12) takes the most compact form

$$H = \frac{1}{2} \left\{ \dot{\varphi}^2 \left[(1 + r \cos(\varphi/2))^2 - \frac{r^2}{4} \cos \varphi \right] + L_0^2 \right\} \quad (15)$$

As usual in the Hamiltonian formulation, it is convenient to introduce

$$\mathbb{J} \equiv \dot{\varphi} = \frac{\left(J + \frac{r L_0 \cos(\varphi/2)}{2} \right)}{\left[(1 + r \cos(\varphi/2))^2 + \frac{r^2}{4} \sin^2(\varphi/2) \right]} \quad (16)$$

then, finally the expression (15) takes the form

$$\begin{aligned} H &= \left\{ \frac{\left(\hat{J} + \frac{r L_0 \cos(\varphi/2)}{2} \right)^2 \left[(1 + r \cos(\varphi/2))^2 - \frac{r^2}{4} \cos \varphi \right]}{\left[(1 + r \cos(\varphi/2))^2 + \frac{r^2}{4} \sin^2(\varphi/2) \right]^2} + L_0^2 \right\} \\ &= \frac{1}{2} \left\{ \mathbb{J}^2 \left[(1 + r \cos(\varphi/2))^2 - \frac{r^2}{4} \cos \varphi \right] + L_0^2 \right\} \end{aligned} \quad (17)$$

These expressions above involving geometry and dynamics on the Möbius strip (MS) will be utilized at the quantum level in next Sections.

IV. ABSTRACT COHERENT STATES

In order to introduce the coherent states for a quantum particle on the Möbius strip geometry we follow the Barut-Girardello construction [2] and we seek the CS as the solution of the eigenvalue equation

$$X |\xi\rangle = \xi |\xi\rangle \quad (18)$$

with complex ξ . Similarly as the standard case where the coherent states $|z\rangle$ satisfy the eigenvalue equation where $z \in \mathbb{C}$:

$$e^{ia} |z\rangle = e^{iz} |z\rangle \quad (19)$$

where a is the standard bosonic annihilation operator with \hat{q} and \hat{p} the position and momentum operators respectively, then we can define

$$X := e^{i(\hat{p} + i\hat{J})} \quad (20)$$

Taking $R = 1$ and inserting (5) into (4) we obtain the parametrization of the band

$$\begin{aligned} X &= \cos\varphi + r \cos(\varphi/2) \cos\varphi \\ Y &= \sin\varphi + r \cos(\varphi/2) \sin\varphi \\ Z &= l + r \sin(\varphi/2) \end{aligned} \quad (21)$$

Taking into account on the initial condition, and the transformations

$$\begin{aligned} X' &= e^{-Z} X \\ Y' &= e^{-Z} Y \\ Z' &= Z \end{aligned} \quad (22)$$

we finally have

$$\xi = e^{-(l+r \sin(\varphi/2)) + i\varphi} (1 + r \cos(\varphi/2)) \quad (23)$$

Inserting above expression in the expansion of the coherent state in the j basis we obtain the CS in explicit form

$$\begin{aligned}
|\xi\rangle &= \sum_{j=-\infty}^{\infty} \xi^{-j} e^{-\frac{j^2}{2}} |j\rangle \\
&= \sum_{j=-\infty}^{\infty} e^{[(l+r \sin(\varphi/2)) - \ln(1+r \cos(\varphi/2)) - i\varphi]j} e^{-\frac{j^2}{2}} |j\rangle \\
&= \sum_{j=-\infty}^{\infty} e^{l'j - i\varphi j} e^{-\frac{j^2}{2}} |j\rangle
\end{aligned} \tag{24}$$

From (24), the fiducial vector is

$$|1\rangle = \sum_{j=-\infty}^{\infty} e^{-\frac{j^2}{2}} |j\rangle \tag{25}$$

then

$$|\xi\rangle = e^{-(\ln\xi)\hat{J}} |1\rangle \tag{26}$$

(in the expression (25) the sum absolutely converges to a finite value $(\Theta_3(0 | e^{-1/2}))$ for $j \in \mathbb{R}$). As it is easily seen the fiducial vector $|1\rangle = |0,0\rangle_{r=0}$ in the (l, φ) parametrization, and this fact permits us rewrite expression (26) as

$$|l, \varphi\rangle = e^{[(l+r \sin(\varphi/2)) - \ln(1+r \cos(\varphi/2)) - i\varphi]j} |0,0\rangle_{r=0} \tag{27}$$

The apparent singularity in (24) corresponding to the case $\xi = 0$ are only for asymptotic values of $(l + r \sin(\varphi/2))$. Notice that in (23) the quantity $(1 + r \cos(\varphi/2))$ never is zero due that $0 < r < R$ with $R = 1$. The overlapping and non-ortogonality formulas are explicitly derived from (26)

$$\langle \xi | \eta \rangle = \sum_{j=-\infty}^{\infty} (\xi^* \eta)^{-j} e^{-j^2} = \Theta_3 \left(\frac{i}{2\pi} \ln(\xi^* \eta) \mid \frac{i}{\pi} \right) \tag{28}$$

and

$$\langle l, \varphi | h, \psi \rangle = \Theta_3 \left(\frac{i}{2\pi} (\varphi - \psi) - \frac{l' + h'}{2} \frac{i}{\pi} \mid \frac{i}{\pi} \right), \tag{29}$$

respectively, where we have been defined l' and h' in order to have more compact expressions as follows:

$$\begin{aligned}
l' &\equiv (l + r \sin(\varphi/2)) - \ln(1 + r \cos(\varphi/2)) \\
h' &\equiv (l + r \sin(\psi/2)) - \ln(1 + r \cos(\psi/2))
\end{aligned}$$

Finally, the normalization as a function of Θ_3 yields

$$\langle \xi | \xi \rangle = \Theta_3 \left(\frac{i}{\pi} \ln |\xi| \mid \frac{i}{\pi} \right) \quad (30)$$

$$\langle l, \varphi | l, \varphi \rangle = \Theta_3 \left(\frac{il'}{\pi} \mid \frac{i}{\pi} \right) \quad (31)$$

V. THE PHYSICAL PHASE SPACE AND THE NATURAL QUANTIZATION

From equations

$$\hat{J}|j\rangle = j|j\rangle \quad (32)$$

$$|l, \varphi\rangle = \sum_{j=-\infty}^{\infty} e^{l'j - i\varphi j} e^{-\frac{j^2}{2}} |j\rangle \quad (33)$$

$$\langle j | l, \varphi \rangle = e^{l'j - i\varphi j} e^{-\frac{j^2}{2}} \quad (34)$$

$$\langle l, \varphi | l, \varphi \rangle = \sum_{j=-\infty}^{\infty} e^{l'j} e^{-j^2} = \Theta_3 \left(\frac{il'}{\pi} \mid \frac{i}{\pi} \right) \quad (35)$$

we notice that the normalization, which for the cylinder (boson case) doesn't depend on φ , depends now on φ through $l' \equiv (l + r \sin(\varphi/2)) - \ln(1 + r \cos(\varphi/2))$. Also

$$\hat{J}|l, \varphi\rangle = \sum_{j=-\infty}^{\infty} e^{l'j - i\varphi j} e^{-\frac{j^2}{2}} j|j\rangle \quad (36)$$

then

$$\frac{\langle \xi | \hat{J} | \xi \rangle}{\langle \xi | \xi \rangle} = \frac{\langle l, \varphi | \hat{J} | l, \varphi \rangle}{\langle l, \varphi | l, \varphi \rangle} = \frac{1}{2\Theta_3 \left(\frac{il'}{\pi} \mid \frac{i}{\pi} \right)} \frac{\partial \Theta_3 \left(\frac{il'}{\pi} \mid \frac{i}{\pi} \right)}{\partial l}. \quad (37)$$

Taking into account the identity

$$\Theta_3 \left(\frac{il'}{\pi} \mid \frac{i}{\pi} \right) = e^{(l')^2} \sqrt{\pi} \Theta_3(l' \mid i\pi) \quad (38)$$

coming from the general formula

$$\Theta_3 \left(\frac{\nu}{\tau} \mid -\frac{1}{\tau} \right) = e^{i\pi\nu^2/\tau} \sqrt{\pi} \Theta_3(\nu \mid \tau) \quad (39)$$

we arrive at the following expression

$$\frac{\langle \xi | \hat{J} | \xi \rangle}{\langle \xi | \xi \rangle} = l' + \frac{1}{2\Theta_3(l' \mid i\pi)} \frac{\partial \Theta_3(l' \mid i\pi)}{\partial l} \quad (40)$$

whichs can be expanded using the following identity for the theta functions

$$\frac{\partial \Theta_3(\nu)}{\partial \nu} = \pi \Theta_3(\nu) \left(\sum_{n=1}^{\infty} \frac{2iq^{2n-1}e^{2i\pi\nu}}{1+q^{2n-1}e^{2i\pi\nu}} - \sum_{n=1}^{\infty} \frac{2iqe^{-2i\pi\nu}}{1+q^{2n-1}e^{-2i\pi\nu}} \right) \quad (41)$$

given explicitly

$$\frac{\langle \xi | \hat{J} | \xi \rangle}{\langle \xi | \xi \rangle} = l' + 2\pi \sin(2l'\pi) \sum_{n=1}^{\infty} \frac{e^{-\pi^2(2n-1)}}{(1+e^{-\pi^2(2n-1)}e^{2i\pi l'}) (1+e^{-\pi^2(2n-1)}e^{-2i\pi l'})} \quad (42)$$

Notice the important result coming from the above expression: the fourth condition required for the CS demands not only l to be integer or semi-integer (as the case for the circle quantization) but also that

$$\varphi = (2k+1)\pi \quad (43)$$

that leads a natural quantization similar as the charge quantization in the Dirac monopole. Precisely this condition over the angle leads the position of the particle in the internal or the external border of the Möbius band, that for $r = \frac{1}{2}$ is $s = \pm \frac{1}{2}$ how is requested to be.

In order to compare our case with the CS constructed in [3] we consider the existence of the unitary operator $U \equiv e^{i\varphi}$, such that $[J, U] = U$ then $U|j\rangle = |j+1\rangle$ such the same average as in the previous case for the \hat{J} operator is:

$$\begin{aligned} \frac{\langle \xi | U | \xi \rangle}{\langle \xi | \xi \rangle} &= e^{-\frac{1}{4}} e^{i\varphi} \frac{\Theta_2\left(\frac{il'}{\pi} \mid \frac{i}{\pi}\right)}{\Theta_3\left(\frac{il'}{\pi} \mid \frac{i}{\pi}\right)} \\ &= e^{-\frac{1}{4}} e^{i\varphi} \frac{\Theta_3(l' + 1/2 \mid i\pi)}{\Theta_3(l' \mid i\pi)} \end{aligned} \quad (45)$$

where in the last equality the relation $\Theta_2(\nu) = e^{i\pi(\frac{1}{4}\tau+\nu)}\Theta_3(\nu + \tau/2)$ was introduced. Also as in [3], we can make the relative average for the operator U in order to eliminate the factor $e^{-\frac{1}{4}}$, then at the first order expression (45) coincides with the unitary circle. It is clear that the denominator in the quotient (45), average with respect to the fiducial CS state, plays the role to centralize the expression of the numerator. However, the claim that U is the best candidate for the position operator is still obscure and requires special analysis that we will be given elsewhere [8].

VI. QUANTUM MECHANICS IN THE MÖBIUS STRIP

The Hamiltonian at quantum level operates as follows

$$\hat{H}|E\rangle = E|E\rangle \quad \text{if} \quad |E\rangle = |j\rangle \rightarrow \quad (46)$$

$$E = \left\{ \frac{\left(j + \frac{rL_0 \cos(\varphi/2)}{2} \right)^2 \left[(1 + r \cos(\varphi/2))^2 - \frac{r^2}{4} \cos \varphi \right]}{\left[(1 + r \cos(\varphi/2))^2 + \frac{r^2}{4} \sin^2(\varphi/2) \right]^2} + L_0^2 \right\} \quad (47)$$

Imposing the fourth requirement, namely $\langle \hat{J} \rangle = l$ for the CS to the expressions, we have $\varphi = (2k+1)\pi$ and the expression (47) for the energy takes the form

$$E = \frac{2j^2}{4 + r^2} + \frac{L_0^2}{2} \quad (48)$$

From the dynamical expressions given above, it is not difficult to make the following remarks:

1) the Hamiltonian is not a priori, T invariant. The H_{MS} is T invariant iff $TL_0 = -L_0$: the variable conjugate to the external momenta l changes under T as J manifesting with this symmetry the full inversion of the motion of the particle on a Möbius strip (evidently it is not the case of the particle motion on the circle).

2) the distribution of energies is Gaussian: from the Bargmann representation [1]

$$\phi_j(\xi^*) \equiv \langle \xi | E \rangle = (\xi^*)^{-j} e^{-\frac{j^2}{2}} \quad (49)$$

the distribution of energies is easily found

$$\frac{|\langle j | \xi \rangle|^2}{\langle \xi | \xi \rangle} = \frac{|\xi|^{-2j} e^{-j^2}}{\Theta_3\left(\frac{i}{\pi} \ln |\xi| \mid \frac{i}{\pi}\right)} = \frac{e^{-2l'j} e^{-j^2}}{\Theta_3\left(\frac{i}{\pi} l' \mid \frac{i}{\pi}\right)}. \quad (50)$$

By the other hand, using the approximate relation from the definition of the Theta function

$$\Theta_3\left(\frac{il'}{\pi} \mid \frac{i}{\pi}\right) = e^{(l')^2} \sqrt{\pi} \left(1 + 2 \sum_{n=1}^{\infty} e^{-\pi^2 n^2} \cos(2l' \pi n) \right) \approx e^{(l')^2} \sqrt{\pi} \quad (51)$$

the expression (50) can be written as

$$\frac{|\langle j | \xi \rangle|^2}{\langle \xi | \xi \rangle} \approx \frac{1}{\sqrt{\pi}} e^{-(j-l')^2} \quad (52)$$

It is useful to remark here that when $\varphi = (2k+1)\pi$ and $l = l'$, the above equation coincides exactly in form with the boson case [3,7] but l is semi-integer valued.

VII. THE PHYSICAL SPACE OF PHASE AND THE PROJECTION METHOD

In order to see how the projection method works in the context of the CS quantization, we start from the torus as our quantum phase space. This means that we have, previous

reduction to the physical phase space via suitable projection operators, $2n$ operators: $\theta, \dot{\theta}, \varphi$ and $\dot{\varphi}$.

$$\begin{aligned} X &= R \cos\varphi + r \sin\theta \cos\varphi \\ Y &= R \sin\varphi + r \sin\theta \sin\varphi \\ Z &= l + r \cos\theta \end{aligned} \quad (53)$$

$$\begin{aligned} \dot{X} &= -\dot{\varphi} \sin\varphi (R + r \sin\theta) + r \cos\theta \cos\varphi \dot{\theta} \\ \dot{Y} &= \dot{\varphi} \cos\varphi (R + r \sin\theta) + r \cos\theta \sin\varphi \dot{\theta} \\ \dot{Z} &= \dot{Z}_0 - r \sin\theta \dot{\theta} \end{aligned} \quad , Z_0 = l \quad (54)$$

Then

$$L_{torus} = \frac{m}{2} \left\{ \dot{\varphi}^2 \left[(R + r \sin\theta)^2 + \frac{r^2}{4} \right] + (r\dot{\theta})^2 - 2r \sin\theta \dot{Z}_0 \dot{\theta} + (\dot{Z}_0)^2 \right\} \quad (55)$$

Before we move to equations of motion of the torus is interesting to notice that inserting the geometrical constraint (5) into the above expression, the Lagrangian of the torus becomes the Lagrangian (7) for the Möbius strip. The Hamiltonian for the torus is easily computed from the following expressions ($m = R = 1$)

$$H = p_\theta \dot{\theta} + p_\varphi \dot{\varphi} + p_{z_0} \dot{Z}_0 - L \quad (56)$$

$$p_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = \dot{\varphi} (1 + r \sin\theta)^2 = J_0 \quad (57)$$

$$p_{z_0} \equiv \frac{\partial L}{\partial \dot{Z}_0} = -r \sin\theta \dot{\theta} + \dot{Z}_0 = L_0 \quad (58)$$

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\theta} - r \sin\theta \dot{Z}_0 \quad (59)$$

$$\begin{aligned} H &= L_{torus} = \frac{1}{2} \left\{ \dot{\varphi}^2 \left[(1 + r \sin\theta)^2 + \frac{r^2}{4} \right] + (r\dot{\theta})^2 - 2r \sin\theta \dot{Z}_0 \dot{\theta} + (\dot{Z}_0)^2 \right\} \\ &= \frac{1}{2} \left\{ \frac{J_0^2}{(1 + r \sin\theta)^2} + \frac{(p_\theta + r \sin\theta L_0)^2}{(r \cos\theta)^2} + L_0^2 \right\} \end{aligned} \quad (60)$$

Now we pass to construct the CS for the torus analogically that in the previous Section for the Möbius strip, but in this case the coordinate θ are absolutely independent of φ . Thus, we assume two "cylinder type" parametrizations: one for $0 \leq l \leq \infty$ cylinder with angular variable φ and the other one with finite $0 \leq l_2 \leq 2\pi \sin^2\varphi (R = 1)$

$$\xi_{torus} = e^{-(l+r \cos\theta)+i\varphi} (1 + r \sin\theta) e^{-2\pi \sin^2\varphi + k\theta} \quad , \quad i^2 = k^2 = -1 \quad (61)$$

We call the above expression the *geometrical* factorization. From the above expression the physical decomposition for $|\xi_{torus}\rangle$ that is useful for our proposal is the following

$$|\xi_{torus}\rangle = \sum_{j,m=-\infty}^{\infty} \xi_{MS}^{-j} e^{-\frac{j^2}{2}} \xi^{-m} e^{-\frac{m^2}{2}} |j, m\rangle \quad (62)$$

$$|\xi_{MS}\rangle = \sum_{j,m=-\infty}^{\infty} \xi_{MS}^{-j} e^{-\frac{j^2}{2}} |j, 0\rangle$$

where we split the part corresponding on the Möbius strip of the rest of the toroidal space of phase

$$\xi_{MS} = e^{-(l-r \sin(\varphi/2)) + \ln(1+r \cos(\varphi/2)) + i\varphi} \quad (63)$$

$$\xi = e^{-2\pi \sin^2 \varphi - r (\cos \theta + \sin(\varphi/2)) + \ln\left(\frac{1+r \sin \theta}{1+r \cos(\varphi/2)}\right) + k\theta}$$

and a m basis was consistently included. This factorization is the *physical* one.

We already have all ingredients to perform the projection from our toroidal phase space to the physical phase space that we are interested in

$$\langle \langle \xi_{MS} | \xi'_{MS} \rangle \rangle = \frac{\langle \xi_{torus} | \xi_{MS} \rangle \langle \xi_{MS} | \xi'_{torus} \rangle}{\langle \xi_{torus}^0 | \xi_{MS} \rangle \langle \xi_{MS} | \xi_{torus}^0 \rangle} = \sum_{j=-\infty}^{\infty} e^{(l'+h')j} e^{-i(\varphi-\psi)} e^{-j^2} \quad (64)$$

with , however, $|\xi_{torus}^0\rangle \equiv |1_{torus}\rangle = \sum_{j,m=-\infty}^{\infty} e^{-\frac{m^2+j^2}{2}} |j, m\rangle$. It is important to note that we can proceed other time performing the projection from the Möbius geometry to the circle straightforwardly obtaining the CS for the Bose case. Then the procedure of projections can be sinthetized in the following schema

$$Torus \rightarrow \text{Projection Op.} \rightarrow \text{Möbius strip}(\text{fermion}) \rightarrow \text{Projection Op.} \rightarrow \text{circle}(\text{boson})$$

Besides the instructive standard procedure given above, where we take advantage on the projection properties of the CS, there exists one powerful method that is based on the universal projector operator

$$E\left(\theta - \frac{\pi + \varphi}{2} \leq \delta\right) = \int_{-\infty}^{\infty} d\lambda e^{-i\left|\theta - \frac{\pi + \varphi}{2}\right|^2} \frac{\sin(\delta^2 \lambda)}{\pi \lambda} \quad (65)$$

that clearly depends only on the constraints, being independent on the specific form of the Hamiltonian or on the form that we factorize the original "big" phase space. For example, it is well known that the CS defined in [2] are a particular case that the CS defined in [6] by means of a displacement operator. This fact is crucial in order to be consistent at the hour to define correctly the observables of the physical system under consideration, in particular the position operator [7,8].

VIII. CONCLUDING REMARKS

In this work the coherent states (CS) for the fermions in the Möbius band was constructed and compared with the previous works where the cylinder was used and the spin $1/2$ was introduced by hand. Using these coherent states particularly constructed we have explicitly shown, that an unoriented surface that is the Möbius band is the natural phase space for fermionic fields. This is because the symmetry properties of the band and the symmetry of the fermions are closely related: both have the characteristic "double covering" that makes that the symmetry invariance is 4π instead the 2π for the bosonic case where the natural phase space is the cylinder. Also because the Coherent states, due the double role that they have, namely, as projectors [10] and making the connection between classical and quantum formulations [5] are very sensibles to the geometrical framework where they was constructed, given the best description of a given physical system. These important facts permit, as we have shown also here, the reduction from the toroidal phase space to the Möbius strip space of phase and lead, due the wonderful proprieties of the CS, a "Dirac-Like" quantization.

It will be interesting to construct coherent states in other geometries and dimensions and to analyze the physical systems that they describe in such cases. This is the main task of future works [8].

IX. ACKNOWLEDGEMENTS

I am very thankful to Professors John Klauder for his advisements and introducing me to the subject of coherent states and projection operators and to E. C. G. Sudarshan for his interest demonstrated in this work in a private communication. Also thanks are given to Professors A. Dorokhov and Yu. Stepanovsky for my scientific formation. This work was partially supported by PNPd-CNPQ brazilian funds.

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